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The influence of fluid inertia on the stability of a plain journal bearing incorporating a complete oil film

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The static equilibrium of a journal rotating within a circular bush in the presence of a complete oil film is considered. Fluid inertia, though having negligible effect on load capacity and attitude angle, is shown to provide an important stabilizing mechanism. For any bearing (with specified geometry and lubricant) there is both a characteristic neutral stability curve and a characteristic operating curve which may intersect in two positions. Analysis of the weakly nonlinear motion, in the vicinity of the neutral curve, reveals the existence of stable, small-amplitude limit cycles which may be either subcritical or supercritical according to where the operating and neutral curves intersect.

1. Introduction

Under normal operating conditions, a journal bearing supports an external load by generating superambient pressures in the convergent section ($0 \leqslant \theta \leqslant \pi$, figure 1) together with an air cavity in the divergent section – arising from the inability of the lubricant to sustain large subambient pressures (Savage 1977; Dowson & Taylor 1979). In such cavitating bearings, fluid inertia is known to be of secondary importance in the determination of load capacity and attitude angle (Pinkus & Sternlicht 1961) and is therefore neglected. In addition Poritsky (1953), Myers (1981) and Gardner *et al.* (1985) have shown that cavitation provides an essential stabilizing mechanism, ensuring the stability of the rotor within the bush over a certain parameter range. The experimental work of Simons (1950) in particular, and also of Cole (1957), suggests that for subambient pressures of sufficiently low magnitude it is possible for a bearing to run in a stable manner with a complete oil film. From a theoretical standpoint, however, Poritsky (1953) and Myers (1981) have used both the long- and short-bearing approximations and the usual assumptions of lubrication theory to show that full-film journal bearings are linearly unstable under all conditions. Clearly there is a paradox to be resolved; once cavitation is absent the principle stabilizing mechanism disappears and the mathematical model needs refinement. In this paper, the effect of fluid inertia is taken into account and shown to provide the means by which stability of a full-film journal bearing can be achieved. It must be stressed, however, that there is no apparent physical reason why inertia should act as a stabilizing (or indeed a destabilizing) mechanism. In addition it may seem surprising that a feature which has negligible effect at zero order (on equilibrium characteristics) may have such a vital effect at first order on stability. Here there is a precedent, as Gardner *et al.* (1984) demonstrated when considering the effect of cavitation boundary conditions on the stability of a journal bearing. They concluded that small differences in boundary conditions, having negligible effect at zero order, can significantly affect the location of the stability borderline.

For any bearing, with specified geometry and lubricant, there is a characteristic neutral curve in (ϵ, ω)-parameter space, where ϵ and ω represent eccentricity and angular running speed respectively. In addition there is a characteristic operating curve, defining the locus of static equilibrium positions, which may intersect the neutral curve in one or two positions. Following Gardner *et al.* (1985), the weakly nonlinear motion of the rotor is analysed once a position of static equilibrium becomes linearly unstable. Results indicate that the qualitative behaviour is quite different when there are two intersections of operating and neutral curves. At one intersection bifurcation may be supercritical (yielding stable small-amplitude limit cycles) or subcritical (yielding unstable limit cycles), whereas at the other the converse applies.

2. Formulation of the model

The basic model consists of a uniform and rigid rotor, subject to an external load $2F$, and supported by two identical journal bearings which are assumed to incorporate a complete (360°) fluid film. The rotor rotates with angular speed ω about its own centre and is assumed to be in a position of static equilibrium. Once such an equilibrium becomes unstable, only symmetric whirling of the rotor is considered, for which each end of the rotor whirls in phase. It is then sufficient to consider one journal bearing only, supporting one half of the total load, as shown in figure 1.

The motion of a Newtonian lubricant of constant density ρ and viscosity μ, having velocity components u and v in the x- and y-directions respectively (where $x = R\theta$ measures distance azimuthally and y across the film) is described by the Navier–Stokes equations. These reduce to the two-dimensional unsteady boundary-layer equations once it is assumed that gradients with respect to y are much greater than those with respect to x, inertia terms appearing only in the x-momentum equation

$$\left.\begin{aligned} \rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right) &= -\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u}{\partial y^2}, \\ 0 &= -\frac{\partial p}{\partial y}, \\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} &= 0, \end{aligned}\right\} \tag{2.1}$$

where p is the fluid pressure. The above formulation, which considers fluid motion in the (x, y)-plane only, is essentially the long-bearing approximation in which flow along the axis of the rotor is neglected.

The motion of the lubricant in the gap, of local film thickness $h(x, t)$, is illustrated in figure 2. The surface of the rotor rotates about its centre A with speed $R\omega$, and appropriate boundary conditions are

$$\left.\begin{aligned} u = v &= 0 && \text{on } y = 0, \\ u = R\omega, \quad v &= R\omega\frac{\partial h}{\partial x}+\frac{\partial h}{\partial t} && \text{on } y = h. \end{aligned}\right\} \tag{2.2}$$

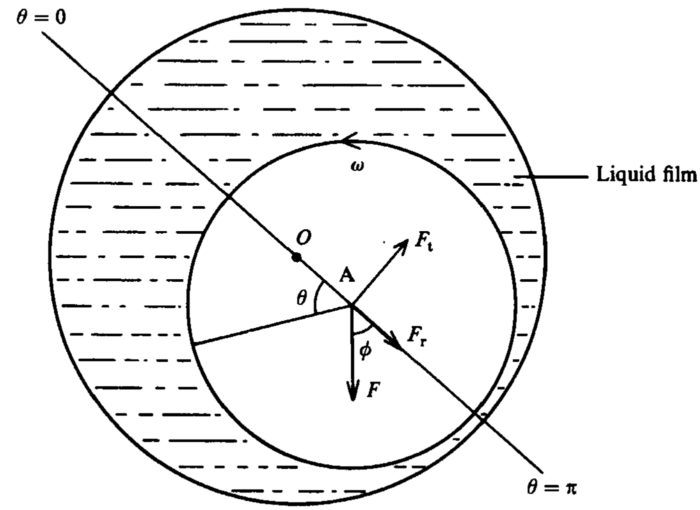

FIGURE 1. A rotor, subject to external load F, operating within a complete liquid film.

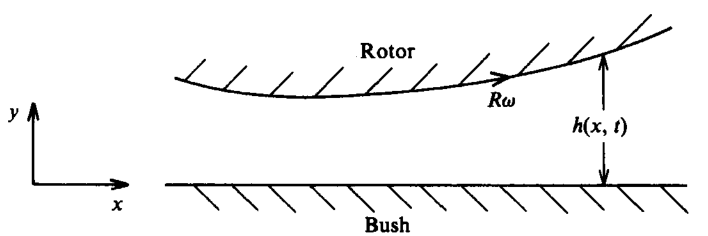

FIGURE 2. A section of the gap between rotor and bush, of local film thickness $h(x, t)$.

Equations (2.1) and (2.2) are normalized by introducing the non-dimensional quantities

$$\theta = \frac{x}{R}, \quad \bar{y} = \frac{y}{c}, \quad \tau = \omega t, \quad \bar{u} = \frac{u}{R\omega}, \quad \bar{v} = \frac{v}{c\omega}, \quad \bar{p} = \frac{p}{\mu\omega(R/c)^2}, \quad \bar{h} = \frac{h}{c}, \tag{2.3}$$

where c is the radial clearance of the bearing. In addition Reynolds number Re and an inertia parameter λ are defined by

$$Re = \frac{\rho c R\omega}{\mu}, \quad \lambda = \frac{c}{R}Re. \tag{2.4}$$

Hence we obtain

$$\left.\begin{aligned} \lambda\left(\frac{\partial \bar{u}}{\partial \tau} + \bar{u}\frac{\partial \bar{u}}{\partial \theta} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}\right) &= -\frac{\partial \bar{p}}{\partial \theta} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}, \\ 0 &= -\frac{\partial \bar{p}}{\partial \bar{y}}, \\ \frac{\partial \bar{u}}{\partial \theta} + \frac{\partial \bar{v}}{\partial \bar{y}} &= 0, \end{aligned}\right\} \tag{2.5}$$

with normalized boundary conditions

$$\left.\begin{aligned} \bar{u} = \bar{v} = 0 \qquad & \text{on } \bar{y} = 0, \\ \bar{u} = 1, \quad \bar{v} = \frac{\partial \bar{h}}{\partial \theta} + \frac{\partial \bar{h}}{\partial \tau} \quad & \text{on } \bar{y} = \bar{h}. \end{aligned}\right\} \tag{2.6}$$

3. Method of solution

In order to solve (2.5) and (2.6) a first-order perturbation expansion in λ is introduced (Reinhardt & Lund 1975):

$$\left.\begin{aligned} \bar{u} &= \bar{u}_0 + \lambda \bar{u}_1 + O(\lambda^2), \\ \bar{v} &= \bar{v}_0 + \lambda \bar{v}_1 + O(\lambda^2), \\ \bar{p} &= \bar{p}_0 + \lambda \bar{p}_1 + O(\lambda^2). \end{aligned}\right\} \tag{3.1}$$

Substituting (3.1) into (2.5) and (2.6) yields, after equating powers of λ, sets of zero- and first-order equations and boundary conditions. The zero-order equations describe the familiar non-inertial model, governed by the Reynolds equation for a long bearing:

$$\frac{\partial}{\partial \theta}\left(\bar{h}^3 \frac{\partial \bar{p}_0}{\partial \theta}\right) = 6\frac{\partial \bar{h}}{\partial \theta} + 12\frac{\partial \bar{h}}{\partial \tau}. \tag{3.2}$$

At first order an inhomogeneous Reynolds equation is obtained for p_1:

$$\begin{aligned} \frac{\partial}{\partial \theta}\left(\bar{h}^3 \frac{\partial \bar{p}_1}{\partial \theta}\right) = \frac{\partial}{\partial \theta}\Bigg\{ & -\frac{3\bar{h}^7}{560}\frac{\partial}{\partial \theta}\left\{\left(\frac{\partial \bar{p}_0}{\partial \theta}\right)^2\right\} - \frac{3\bar{h}^6}{140}\frac{\partial \bar{h}}{\partial \theta}\left(\frac{\partial \bar{p}_0}{\partial \theta}\right)^2 + \frac{\bar{h}^5}{20}\frac{\partial^2 \bar{p}_0}{\partial \theta^2} \\ & + \frac{13\bar{h}^4}{140}\frac{\partial \bar{h}}{\partial \theta}\frac{\partial \bar{p}_0}{\partial \theta} - \frac{\bar{h}^2}{10}\frac{\partial \bar{h}}{\partial \theta} + \frac{13\bar{h}^4}{70}\frac{\partial \bar{p}_0}{\partial \theta}\frac{\partial \bar{h}}{\partial \tau} + \frac{\bar{h}^5}{10}\frac{\partial^2 \bar{p}_0}{\partial \theta \partial \tau}\Bigg\}. \end{aligned} \tag{3.3}$$

The boundary conditions on the pressure for a complete film are

$$\bar{p}(0) = \bar{p}(2\pi) = 0. \tag{3.4}$$

Non-dimensional hydrodynamic force components $\bar{F}_r$ and $\bar{F}_t$, acting along the radial and tangential directions respectively, are determined via

$$\left.\begin{aligned} \bar{F}_r &= \int_0^{2\pi} \bar{p}(\theta)\cos\theta \, d\theta, \\ \bar{F}_t &= \int_0^{2\pi} \bar{p}(\theta)\sin\theta \, d\theta, \end{aligned}\right\} \tag{3.5}$$

which, using (3.1) and (3.4), may be written as

$$\left.\begin{aligned} \bar{F}_r &= -\int_0^{2\pi} \frac{\partial \bar{p}_0}{\partial \theta}\sin\theta \, d\theta - \lambda\int_0^{2\pi} \frac{\partial \bar{p}_1}{\partial \theta}\sin\theta \, d\theta + O(\lambda^2), \\ \bar{F}_t &= \int_0^{2\pi} \frac{\partial \bar{p}_0}{\partial \theta}\cos\theta \, d\theta + \lambda\int_0^{2\pi} \frac{\partial \bar{p}_1}{\partial \theta}\cos\theta \, d\theta + O(\lambda^2). \end{aligned}\right\} \tag{3.6}$$

The non-dimensional film thickness for a journal bearing with a small clearance ratio (c/R) is well approximated by

$$\bar{h} = 1 + \epsilon\cos\theta + O\left(\frac{c}{R}\right), \tag{3.7}$$

where the eccentricity, $e (= c\epsilon)$, is the distance between A and O (figure 1), the centres of the rotor and bush respectively. Utilizing (3.7), equations (3.2) and (3.3) are solved subject to (3.4) to give expressions for the zero- and first-order pressure gradients. Hence (3.6) yield

$$\begin{aligned} \bar{F}_{\mathrm{r}} = {} & \frac{-12\pi\epsilon'}{(1-\epsilon^2)^{\frac{3}{2}}} - \lambda\pi\left\{\frac{3\epsilon(1-2\phi')^2}{35(2+\epsilon^2)^2}\{3(2+\epsilon^2)-(1-\epsilon^2)^{\frac{1}{2}}(20+\epsilon^2)\}\right. \\ & \left. + \frac{\{1-(1-\epsilon^2)^{\frac{1}{2}}\}}{5\epsilon} + \frac{12\epsilon''}{5\epsilon^2}\{1-(1-\epsilon^2)^{\frac{1}{2}}\} - \frac{6\epsilon'^2}{5\epsilon^3}\left\{2-\frac{(2-\epsilon^2)}{(1-\epsilon^2)^{\frac{1}{2}}}\right\}\right\} + O(\lambda^2), \\ \bar{F}_{\mathrm{t}} = {} & \frac{12\pi(1-2\phi')\,\epsilon}{(2+\epsilon^2)(1-\epsilon^2)^{\frac{1}{2}}} + \lambda\pi\left\{\frac{6\epsilon'(1-2\phi')}{35(2+\epsilon^2)^3}\{3(26-\epsilon^2)(2+\epsilon^2)+(1-\epsilon^2)^{\frac{1}{2}}(\epsilon^4+36\epsilon^2-100)\}\right. \\ & \left. + \frac{12\phi''}{5\epsilon(2+\epsilon^2)^2}\{(2+\epsilon^2)(2-5\epsilon^2)-4(1-\epsilon^2)^{\frac{3}{2}}\}\right\} + O(\lambda^2), \end{aligned} \tag{3.8}$$

where the attitude angle ϕ is the angle between the line of centres OA and the direction of the applied load (figure 1), and a prime denotes differentiation with respect to τ.

In the above analysis the non-dimensional film thickness is approximated by (3.7) in which curvature effects, of $O(c/R)$, are omitted. Since $\lambda = (c/R)\,Re$, the inclusion of inertia is strictly valid only for Re much greater than unity. Such a restriction on Re is in fact unnecessary from the standpoint of locating the neutral stability curve – as discussed in §4, (4.20) onwards.

We note also that the effect of inertia is to introduce extra terms into the expressions (3.8) for $\bar{F}_{\mathrm{r}}$ and $\bar{F}_{\mathrm{t}}$, the presence of which permits linear stability in some parameter range. This conclusion that inertia, like cavitation, provides a stabilizing mechanism follows via the mathematics, yet in each case no obvious physical explanation is apparent.

4. Linear stability analysis

For considering the stability of the rotor it is appropriate to introduce a Cartesian coordinate system as shown in figure 3, where

$$X = \epsilon\cos\phi, \quad Y = \epsilon\sin\phi. \tag{4.1}$$

Non-dimensional force components in the X- and Y-directions are given by $\bar{F}_x$ and $\bar{F}_y$ respectively

$$\left.\begin{aligned} \bar{F}_x &= \bar{F}_{\mathrm{r}}\cos\phi - \bar{F}_{\mathrm{t}}\sin\phi + \frac{1}{S}, \\ \bar{F}_y &= \bar{F}_{\mathrm{r}}\sin\phi + \bar{F}_{\mathrm{t}}\cos\phi, \end{aligned}\right\} \tag{4.2}$$

where S is the Sommerfeld number defined by

$$S = \frac{LR^3\omega\mu}{Fc^2}, \tag{4.3}$$

and L is the axial length of the bearing.

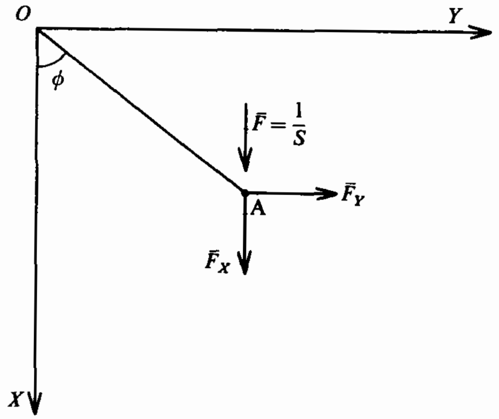

FIGURE 3. Non-dimensional Cartesian coordinate system with origin O, the centre of the bush.

The non-dimensional equations of motion include force components which are nonlinear in X, Y, X', Y', X'', Y'' and take full account of inertia:

$$\left.\begin{aligned} X'' &= \frac{S}{\bar{\omega}^2}\bar{F}_X(X, Y, X', Y', X'', Y''), \\ Y'' &= \frac{S}{\bar{\omega}^2}\bar{F}_Y(X, Y, X', Y', X'', Y''), \end{aligned}\right\} \tag{4.4}$$

where $\bar{\omega}$ is a non-dimensional rotor speed

$$\bar{\omega} = \left(\frac{mc}{F}\right)^{\frac{1}{2}}\omega \tag{4.5}$$

and $2m$ is the mass of the rotor.

The linear stability of a static equilibrium position (X_s, Y_s) is found by writing

$$x = X - X_s, \quad y = Y - Y_s \tag{4.6}$$

and expanding the right-hand side of (4.4) as a first-order Taylor series about (X_s, Y_s), subscript s denoting a steady state. Hence

$$\bar{F}_{Xs} = 0, \quad \bar{F}_{Ys} = 0 \tag{4.7}$$

and

$$\left.\begin{aligned} \bar{\omega}^2 x'' + K_{XX}x + K_{XY}y + B_{XX}x' + B_{XY}y' + C_{XX}x'' + C_{XY}y'' &= 0, \\ \bar{\omega}^2 y'' + K_{YX}x + K_{YY}y + B_{YX}x' + B_{YY}y' + C_{YX}x'' + C_{YY}y'' &= 0, \end{aligned}\right\} \tag{4.8}$$

where

$$\left.\begin{aligned} K_{XX} &= -S\left(\frac{\partial \bar{F}_X}{\partial X}\right)_s, \quad K_{YX} = -S\left(\frac{\partial \bar{F}_Y}{\partial X}\right)_s, \\ K_{XY} &= -S\left(\frac{\partial \bar{F}_X}{\partial Y}\right)_s, \quad K_{YY} = -S\left(\frac{\partial \bar{F}_Y}{\partial Y}\right)_s \end{aligned}\right\} \tag{4.9a}$$

are displacement coefficients,

$$\left.\begin{aligned} B_{XX} &= -S\left(\frac{\bar{F}_X}{\partial X'}\right)_s, \quad B_{YX} = -S\left(\frac{\partial \bar{F}_Y}{\partial X'}\right)_s, \\ B_{XY} &= -S\left(\frac{\partial \bar{F}_X}{\partial Y'}\right)_s, \quad B_{YY} = -S\left(\frac{\partial \bar{F}_Y}{\partial Y'}\right)_s \end{aligned}\right\} \tag{4.9b}$$

are velocity coefficients, and

$$\left.\begin{aligned} C_{XX} &= -S\left(\frac{\partial \bar{F}_X}{\partial \overset{\prime\prime}{X}}\right)_s, \quad C_{YX} = -S\left(\frac{\partial \bar{F}_Y}{\partial \overset{\prime\prime}{X}}\right)_s, \\ C_{XY} &= -S\left(\frac{\partial \bar{F}_X}{\partial \overset{\prime\prime}{Y}}\right)_s, \quad C_{YY} = -S\left(\frac{\partial \bar{F}_Y}{\partial \overset{\prime\prime}{Y}}\right)_s \end{aligned}\right\} \tag{4.9c}$$

are acceleration coefficients.

This procedure, though formally correct, is nevertheless impractical since the evaluation of coefficients K_{XX}, etc., to order λ, requires that derivatives of the force coefficients are known to order λ, e.g.

$$\frac{\partial \bar{F}_X}{\partial X} = F^0 + \lambda F' + O(\lambda^2). \tag{4.10}$$

This is not the case, and indeed the only way to proceed is to start with the force coefficients expanded as an asymptotic series in λ,

$$\bar{F}_X = F^0_X + \lambda F'_X + O(\lambda^2), \tag{4.11}$$

each term of which is then expanded in a Taylor series about (X_s, Y_s). Essentially it is assumed – without proof – that the double expansion of $\bar{F}_X$ and $\bar{F}_Y$, as a Taylor series and as a series in λ, can be effected irrespective of order.

The following observations can now be made:

(i) Parameters λ and S are not independent since both are directly proportional to rotor speed. Hence we write

$$\lambda = \sigma_1 S, \tag{4.12}$$

where σ_1 is a system parameter defined by

$$\sigma_1 = \frac{F\rho c^4}{LR^3\mu^2}, \tag{4.13}$$

such that for any bearing with specified geometry, lubricant and external load, σ_1 is a constant.

(ii) The steady state solution of (4.4) is described by (4.7) which, via (4.2), yields

$$\left.\begin{aligned} \cos\phi_s &= -S\bar{F}_{rs}, \\ \sin\phi_s &= S\bar{F}_{ts}. \end{aligned}\right\} \tag{4.14}$$

Furthermore, the force coefficients have been determined to order λ (equations (3.8)) such that

$$\left.\begin{aligned} \bar{F}_{rs} &= \lambda F_1(\epsilon_s) + O(\lambda^2), \\ \bar{F}_{ts} &= F_2(\epsilon_s) + O(\lambda^2), \end{aligned}\right\} \tag{4.15}$$

where F_1 and F_2 are functions of ϵ_s only. Equations (4.14) and (4.15) now yield

$$\left.\begin{aligned} \cos\phi_s &= -S(\lambda F_1 + O(\lambda^2)), \\ \sin\phi_s &= S(F_2 + O(\lambda^2)). \end{aligned}\right\} \tag{4.16}$$

At this stage we note that all physical solutions of (4.16) require λ to be strictly positive. These include the 'inertialess lubrication solution' which, via (4.12), corresponds to (4.16) correct to order λ; $\sigma_1 \cos\phi_s = 0$, $\sigma_1 \sin\phi_s = \lambda F_2$. Consequently

$\phi_s = \frac{1}{2}\pi$, $\lambda = \sigma_1/F_2$ and the Sommerfeld number S is given by the familiar expression

$$S = \frac{1}{F_2(\epsilon_s)} = \frac{(2+\epsilon_s^2)(1-\epsilon_s^2)^{\frac{1}{2}}}{12\pi\epsilon_s}. \tag{4.17}$$

With inertia taken into account, (4.17) still remains valid to order λ since (4.16) yield

$$S = \frac{1}{F_2}(1+O(\lambda^2)). \tag{4.18}$$

Seeking solutions to (4.8) of the form

$$x = x_0\,e^{\bar{n}\tau}, \quad y = y_0\,e^{\bar{n}\tau} \tag{4.19}$$

leads to a characteristic equation of fourth degree in $\bar{n}$:

$$\begin{aligned}&\{\bar{\omega}^4 + (C_{XX}+C_{YY})\bar{\omega}^2 + (C_{XX}C_{YY}-C_{XY}C_{YX})\}\bar{n}^4\\&+\{(B_{XX}+B_{YY})\bar{\omega}^2+(B_{XX}C_{YY}+B_{YY}C_{XX}-B_{XY}C_{YX}-B_{YX}C_{XY})\}\bar{n}^3\\&+\{(K_{XX}+K_{YY})\bar{\omega}^2+(K_{XX}C_{YY}+K_{YY}C_{XX}-K_{XY}C_{YX}-K_{YX}C_{XY}\\&+B_{XX}B_{YY}-B_{XY}B_{YX})\}\bar{n}^2\\&+\{K_{XX}B_{YY}+K_{YY}B_{XX}-K_{XY}B_{YX}-K_{YX}B_{XY}\}\bar{n}\\&+\{K_{XX}K_{YY}-K_{XY}K_{YX}\} = 0.\end{aligned} \tag{4.20}$$

The stability criterion of Liénard & Chipart (see Appendix A) is used to examine the roots of (4.20) and in particular, with σ_1 specified, a critical value of angular speed $\bar{\omega}_c$ is determined for each value of eccentricity ϵ_s such that for $\bar{\omega} < \bar{\omega}_c(\bar{\omega} > \bar{\omega}_c)$, the rotor is linearly stable (unstable). Neutral stability curves in $(\epsilon_s, \bar{\omega})$-parameter space are shown in figure 4, from which it is instructive to note that for each journal bearing with system parameter σ_1 there is a characteristic neutral curve, and all such neutral curves reduce, in the absence of inertia, to the same curve, $\bar{\omega} = 0$.

Application of the stability criterion reveals that $\bar{\omega}_c \sim \lambda^{\frac{1}{2}}$ (see Appendix A) which, via (4.12) and (4.18), appears to imply that the neutral curves are asymptotic to the $\bar{\omega}$-axis. However, employment of the perturbation expansion (3.1) necessitates that λ is small, thus invalidating this implication. Indeed the dotted lines shown in figure 4 indicate the values of ϵ_s below which λ exceeds 0.1; $\epsilon_s = 0.05$ (0.26) for $\sigma_1 = 0.1$ (0.5).

In the preceding analysis the inclusion of inertia terms of $O(\lambda)$, where $\lambda = (c/R)\,Re$, and the omission of curvature terms of $O(c/R)$, strictly requires that $Re \gg 1$. However, such a strong condition is unnecessary since the effect of curvature on displacement, velocity and acceleration coefficients (equation (4.9)) and on the corresponding location of the neutral stability curve has been fully investigated and found to be negligible. We may therefore conclude that, with Re of order 1, curvature effects are unimportant and may be ignored – in sharp contrast to $O(c/R)$ inertia terms, which significantly affect the stability characteristics.

Clearly the introduction of the system parameter σ_1 is to facilitate the determination of the neutral stability curve. For any given journal bearing however, a second system parameter σ_2 must be introduced in order to plot the locus of static equilibrium positions, and for this we write

$$S(\epsilon_s) = \sigma_2\,\bar{\omega}, \tag{4.21}$$

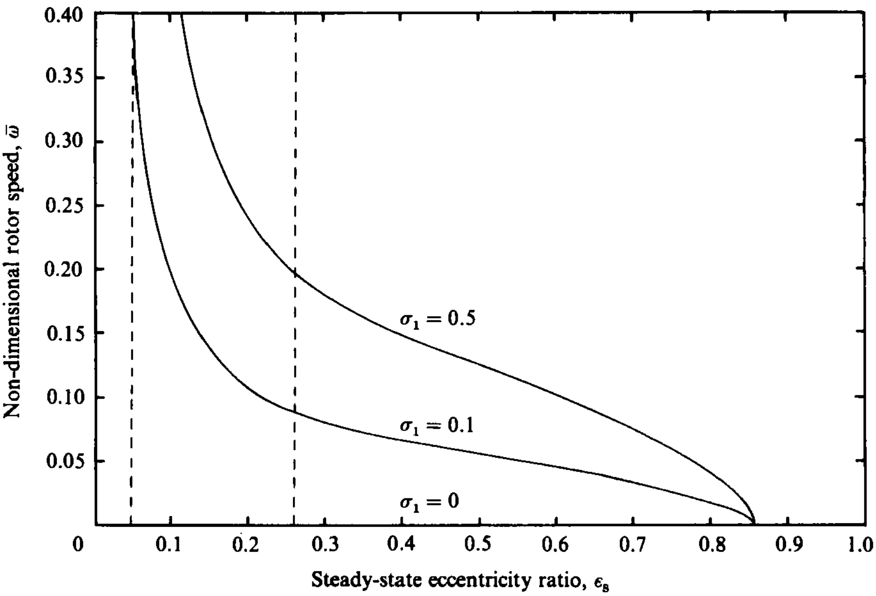

FIGURE 4. Neutral stability curves ($\sigma_1 = 0, 0.1, 0.5$).

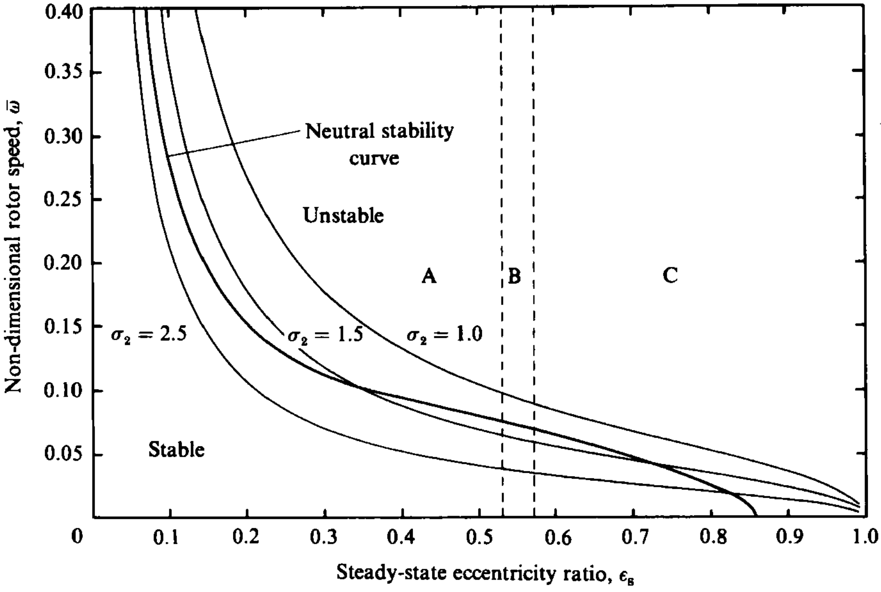

FIGURE 5. Neutral stability curve ($\sigma_1 = 0.2$) with three operating curves ($\sigma_2 = 1.0, 1.5, 2.5$).

which, via (4.3) and (4.5), yields

$$\sigma_2 = \frac{LR^3\mu}{(Fmc^5)^{\frac{1}{2}}}. \tag{4.22}$$

Consequently any given journal bearing has two system parameters, σ_1 and σ_2, both of which are constant. Equation (4.21) enables an operating curve to be drawn representing the locus of equilibrium states. Figure 5 shows the neutral curve for a particular value of σ_1 and includes operating curves for three values of σ_2, illustrating how intersection may arise in none, one or two positions. At each point of intersection conditions are critical; $\bar{\omega} = \bar{\omega}_c$ and two of the roots of (4.20) are purely imaginary, such that if $\bar{n}$ is expressed in the form

$$\bar{n}(\bar{\omega}) = \bar{\alpha}(\bar{\omega}) \pm \mathrm{i}\bar{\Omega}(\bar{\omega}), \tag{4.23}$$

then

$$\bar{\alpha}(\bar{\omega}_c) = 0, \quad \bar{\Omega}(\bar{\omega}_c) = \bar{\Omega}_c. \tag{4.24}$$

5. Method of multiple scales

The weakly nonlinear motion of a rotor in a journal bearing with cavitation (modelled by a 180° fluid film) has been analysed by Gardner *et al.* (1985) using the method of multiple scales. For a complete fluid film with inertia included the same method can be applied in a similar manner, the only difference being a substantial increase in algebra, since the force components $(\bar{F}_X, \bar{F}_Y)$ are now functions of six arguments (equation (4.4)). Hence only a brief outline of the method is given, with the details in Appendix B.

Equation (4.6) defines $x(\tau)$ and $y(\tau)$, both of which are assumed to be small, and hence the equations of motion (4.4) may be expanded about an equilibrium position (X_s, Y_s) as far as third order:

$$\begin{aligned}\bar{\omega}^2 x'' = {} & a_1 x + a_2 y + a_3 x' + a_4 y' + a_5 x'' + a_6 y'' \\ & + \tfrac{1}{2}a_{11} x^2 + a_{12} xy + a_{13} xx' + a_{14} xy' + a_{15} xx'' + a_{16} xy'' \\ & + \tfrac{1}{2}a_{22} y^2 + \ldots + \tfrac{1}{2}a_{66} y''^2 \\ & + \tfrac{1}{6}a_{111} x^3 + \tfrac{1}{2}a_{112} x^2 y + \tfrac{1}{2}a_{113} x^2 x' + \tfrac{1}{2}a_{114} x^2 y' + \tfrac{1}{2}a_{115} x^2 x'' + \tfrac{1}{2}a_{116} x^2 y'' \\ & + \tfrac{1}{2}a_{122} xy^2 + a_{123} xyx' + \ldots + \tfrac{1}{6}a_{666} y''^3, \end{aligned} \tag{5.1}$$

where the a_i, a_{ij}, a_{ijk} $(1 \leqslant i,j,k \leqslant 6)$ are defined in Appendix B. There is a similar expression for $\bar{\omega}^2 y''$ with the a_i, etc. replaced by b_i, etc.

Close to the neutral curve, supercritical bifurcation is considered, for which we introduce the small, real parameter δ defined by

$$\delta^2 = \bar{\omega} - \bar{\omega}_c, \tag{5.2}$$

and in addition slow and fast timescales defined by

$$\tau^* = \delta^2 \tau, \tag{5.3}$$

$$s = (1 + \delta\omega_1 + \delta^3\omega_3 + \ldots)\,\tau. \tag{5.4}$$

Displacements $x(\tau)$, $y(\tau)$ are also expanded in powers of δ:

$$x(s, \tau^*) = \delta x_1(s, \tau^*) + \delta^2 x_2(s, \tau^*) + \delta^3 x_3(s, \tau^*) + \ldots, \tag{5.5}$$

$$y(s, \tau^*) = \delta y_1(s, \tau^*) + \delta^2 y_2(s, \tau^*) + \delta^3 y_3(s, \tau^*) + \ldots, \tag{5.6}$$

where the ω_i, x_i and y_i are as yet unknown functions. Denoting time differentiating with respect to s by $(\cdot)$, then

$$x' = \delta \dot{x}_1 + \delta^2(\dot{x}_2 + \omega_1 \dot{x}_1) + \delta^3\left(\dot{x}_3 + \omega_1 \dot{x}_2 + \frac{\partial x_1}{\partial \tau^*}\right), \tag{5.7}$$

$$x'' = \delta \ddot{x}_1 + \delta^2(\ddot{x}_2 + 2\omega_1 \ddot{x}_1) + \delta^3\left(\ddot{x}_3 + 2\omega_1 \ddot{x}_2 + \omega_1^2 \ddot{x}_1 + 2\frac{\partial \dot{x}_1}{\partial \tau^*}\right), \tag{5.8}$$

and hence corresponding powers of δ may be equated in (5.1) to yield

Order δ

$$\ddot{x}_1-\bar{a}_1x_1-\bar{a}_2y_1-\bar{a}_3\dot{x}_1-\bar{a}_4\dot{y}_1-\bar{a}_5\ddot{x}_1-\bar{a}_6\ddot{y}_1=0, \tag{5.9a}$$

Order δ^2

$$\begin{aligned}\ddot{x}_2-\bar{a}_1x_2-\bar{a}_2y_2-\bar{a}_3\dot{x}_2-\bar{a}_4\dot{y}_2-\bar{a}_5\ddot{x}_2-\bar{a}_6\ddot{y}_2 \\ =-2\omega_1\ddot{x}_1+\bar{a}_3\omega_1\dot{x}_1+\bar{a}_4\omega_1\dot{y}_1+2\bar{a}_5\omega_1\ddot{x}_1+2\bar{a}_6\omega_1\ddot{y}_1+\tfrac{1}{2}\bar{a}_{11}x_1^2+\bar{a}_{12}x_1y_1 \\ +\bar{a}_{13}x_1\dot{x}_1+\bar{a}_{14}x_1\dot{y}_1+\bar{a}_{15}x_1\ddot{x}_1+\bar{a}_{16}x_1\ddot{y}_1+\tfrac{1}{2}\bar{a}_{22}y_1^2+\bar{a}_{23}y_1\dot{x}_1+\bar{a}_{24}y_1\dot{y}_1 \\ +\bar{a}_{25}y_1\ddot{x}_1+\bar{a}_{26}y_1\ddot{y}_1+\tfrac{1}{2}\bar{a}_{33}\dot{x}_1^2+\bar{a}_{34}\dot{x}_1\dot{y}_1+\tfrac{1}{2}\bar{a}_{44}\dot{y}_1^2,\end{aligned} \tag{5.10a}$$

where
$$\bar{a}_i=\frac{a_i}{\bar{\omega}_c^2},\quad \bar{a}_{ij}=\frac{a_{ij}}{\bar{\omega}_c^2},\quad \bar{a}_{ijk}=\frac{a_{ijk}}{\bar{\omega}_c^2}. \tag{5.11}$$

Two additional equations, (5.9*b*) and (5.10*b*), are obtained by considering the equation of motion in the Y-direction, whilst at order δ^3 two further equations, (B 1), emerge, details of which are to be found in Appendix B.

Neglecting transients, the solution to (5.9) is

$$\left.\begin{aligned}x_1(s,\tau^*)&=A_1(\tau^*)\,e^{i\bar{\Omega}_c s}+\text{c.c.},\\ y_1(s,\tau^*)&=\beta A_1(\tau^*)\,e^{i\bar{\Omega}_c s}+\text{c.c.},\end{aligned}\right\} \tag{5.12}$$

where β is a complex constant dependent on ϵ_s only and c.c. denotes the complex-conjugate of the preceding expression. Substituting the above expressions into the right-hand sides of (5.10), the suppression of secular terms requires

$$\omega_1=0, \tag{5.13}$$

and the right-hand sides of (5.10) can thus be written in the form

$$\left.\begin{aligned}&A_1^2m_1\,e^{2i\bar{\Omega}_c s}+|A_1|^2\tfrac{1}{2}n_1+\text{c.c.},\\ &A_1^2m_2\,e^{2i\bar{\Omega}_c s}+|A_1|^2\tfrac{1}{2}n_2+\text{c.c.},\end{aligned}\right\} \tag{5.14}$$

were m_1, m_2 are complex constants, n_1, n_2 real constants and all are dependent on ϵ_s only. The solution to (5.10) now follows:

$$\left.\begin{aligned}x_2(s,\tau^*)&=A_2(\tau^*)\,e^{i\bar{\Omega}_c s}+u_1A_1^2(\tau^*)\,e^{2i\bar{\Omega}_c s}+\tfrac{1}{2}v_1|A_1(\tau^*)|^2+\text{c.c.},\\ y_2(s,\tau^*)&=\beta A_2(\tau^*)\,e^{i\bar{\Omega}_c s}+u_2A_1^2(\tau^*)\,e^{2i\bar{\Omega}_c s}+\tfrac{1}{2}v_2|A_1(\tau^*)|^2+\text{c.c.},\end{aligned}\right\} \tag{5.15}$$

were u_1, u_2 are complex constants, v_1, v_2 real constants and all are dependent on ϵ_s only. We now consider equations (B 1) and suppress secular terms from the presence of $e^{i\bar{\Omega}_c s}$ and $e^{-i\bar{\Omega}_c s}$ on the right-hand sides. The coefficients of $e^{i\bar{\Omega}_c s}$ are of the form

$$\left.\begin{aligned}\gamma_1\frac{dA_1}{d\tau^*}+A_1(\gamma_2+\gamma_3|A_1|^2)&=q_1,\\ \gamma_4\frac{dA_1}{d\tau^*}+A_1(\gamma_5+\gamma_6|A_1|^2)&=q_2\end{aligned}\right\} \tag{5.16}$$

where the γ's are obtained by substitution of x_1, x_2, y_1 and y_2 into the right-hand side of (B 1). The corresponding coefficients of $e^{-i\bar{\Omega}_c s}$ are $\bar{q}_1$ and $\bar{q}_2$.

Particular integral solutions to (B 1) for $x_3(t)$ include a secular term $s e^{i\bar{\Omega}_c s}$, the coefficient of which is of the form $B_1q_1+B_2q_2$, B_1, B_2 being functions of ϵ_s only.

Similarly the coefficient of $se^{i\Omega_c s}$ in the solution for $y_3(t)$ is found to be a multiple of $(B_1 q_1 + B_2 q_2)$. Hence we require

$$B_1 q_1 + B_2 q_2 = 0, \tag{5.17}$$

which reduces, on substituting for q_1 and q_2, to the following complex amplitude equation:

$$\frac{dA_1}{d\tau^*} = A_1(\eta_2 - \eta_3 |A_1|^2), \tag{5.18}$$

η_2, η_3 being functions of ϵ_s only. By writing

$$\left.\begin{aligned} A_1(\tau^*) &= R(\tau^*)\, e^{i\theta(\tau^*)}, \\ \eta_2 = \eta_{2r} + i\eta_{2i}, &\quad \eta_3 = \eta_{3r} + i\eta_{3i}, \end{aligned}\right\} \tag{5.19}$$

and separating out real and imaginary parts, the following amplitude and phase shift equations are derived:

$$\frac{dR}{d\tau^*} = R(\eta_{2r} - \eta_{3r} R^2), \tag{5.20}$$

$$\frac{d\theta}{d\tau^*} = \eta_{2i} - \eta_{3i} R^2, \tag{5.21}$$

where η_{2r} represents the linear growth rate $(d\bar{\alpha}/d\bar{\omega})_{\bar{\omega}=\bar{\omega}_c}$, which is evaluated on the stability borderline and is positive as the rotor moves along an operating curve of constant σ_2 from a stable to an unstable position.

The analysis of subcritical bifurcation requires the replacement of (5.2) by

$$\delta^2 = \bar{\omega}_c - \bar{\omega} \tag{5.22}$$

and an amplitude equation similar to (5.20) is obtained, with the coefficient of R now of opposite sign,

$$\frac{dR}{d\tau^*} = -R(\eta_{2r} + \eta_{3r} R^2). \tag{5.23}$$

6. Discussion of results and suggestions for experimental work

Figure 5 illustrates the neutral stability curve for journal bearings having system parameter $\sigma_1 = 0.2$. The coefficients η_{2r} and η_{3r} are evaluated at various points on the curve, and (5.20) and (5.23) are used to examine the weakly nonlinear behaviour at both supercritical and subcritical speeds. Parameter space is divided into three distinct regions A, B and C as shown in figure 5.

Region A: $(0 < \epsilon_s \leqslant 0.53)$ $\quad \eta_{2r} > 0, \quad \eta_{3r} < 0.$

For $\bar{\omega} > \bar{\omega}_c$, (5.20) implies that $dR/d\tau^* > 0$ for all τ^* and so there is no evolution towards a periodic orbit. For $\bar{\omega} < \bar{\omega}_c$, a periodic orbit is theoretically possible with $R^2(\tau^*) = -\eta_{2r}/\eta_{3r}$ provided $R^2(0) = -\eta_{2r}/\eta_{3r}$. Such a motion is, however, unstable via (5.23).

Region B: $(0.53 < \epsilon_s < 0.57)$ $\quad \eta_{2r} > 0, \quad \eta_{3r} > 0.$

For $\bar{\omega} > \bar{\omega}_c$, (5.20) implies the existence of a stable periodic orbit with amplitude proportional to $(\eta_{2r}/\eta_{3r})^{\frac{1}{2}}$, whilst for $\bar{\omega} < \bar{\omega}_c$, $R(\tau^*) \to 0$ as $\tau^* \to \infty$ via (5.23).

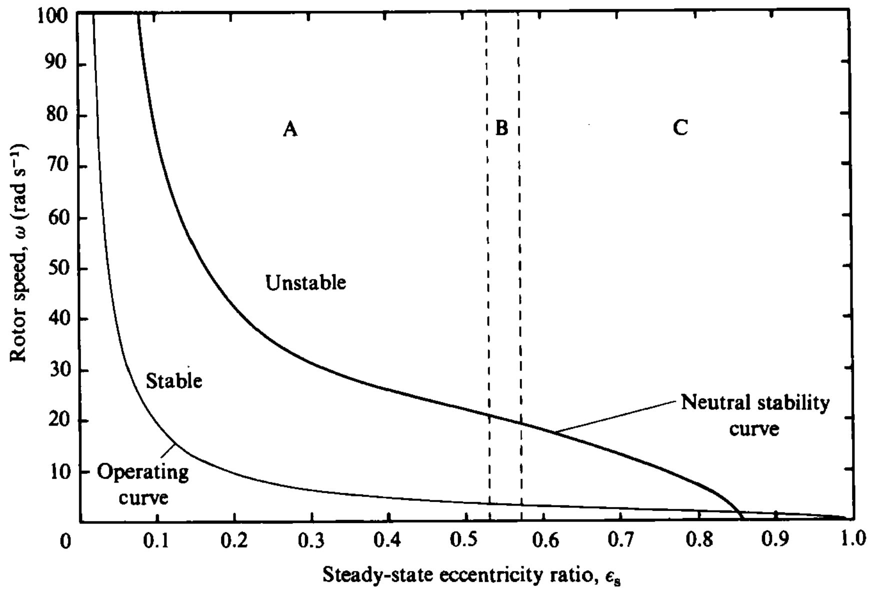

FIGURE 6. Neutral stability and operating curves ($\sigma_1 = 0.0788$, $\sigma_2 = 12.3$).

Region C: $(0.57 \leqslant \epsilon_s < 1.0)$ $\quad \eta_{2r} < 0, \quad \eta_{3r} > 0.$

For $\bar{\omega} > \bar{\omega}_c$, $R(\tau^*) \to 0$ as $\tau^* \to \infty$ via (5.21), whilst for $\bar{\omega} < \bar{\omega}_c$, (5.23) implies the existence of a stable periodic orbit with amplitude proportional to $(-\eta_{2r}/\eta_{3r})^{\frac{1}{2}}$.

Regions A, B and C can be characterized respectively as regions of subcritical (unstable) bifurcation, supercritical (stable) bifurcation and subcritical (stable) bifurcation. Gardner *et al.* (1985) identified regions analogous to A and B but did not find a region corresponding to C. Indeed, region C is where $(\mathrm{d}\bar{\alpha}/\mathrm{d}\bar{\omega})_{\bar{\omega}=\bar{\omega}_c} < 0$, which does not occur for non-inertial, cavitating journal bearings since the growth rate of small disturbances is positive along the full length of the neutral curve. The presence of region C makes for the interesting possibility of discovering small-amplitude limit cycles at low speeds and high eccentricities.

With a view to experimental confirmation of the key results emerging from this theory, namely (i) the presence and extent of a region of parameter space where positions of static equilibrium are to be found, and (ii) the existence of subcritical (stable) small-amplitude limit cycles close to the stability borderline, two bearing systems are now examined. Both consist of a solid steel shaft of length 0.3 m and diameter $(2R)$ 0.1 m (mass $2m = 18.4$ kg), supported symmetrically in two bearings $(F = 90$ N) of length-to-diameter ratio 0.5 $(L = 0.05$ m). With lubricant density ρ taken as 875 kg m^{-3}, the two systems are characterized by clearance ratio c and lubricant dynamic viscosity μ, giving rise to particular values of σ_1 and σ_2:

$$\text{(i)}\ \frac{c}{R} = \frac{1}{1000} \quad (c = 5\times10^{-5}\ \text{m}), \quad \mu = 0.001\ \text{N s m}^{-2}; \quad \sigma_1 = 0.0788, \quad \sigma_2 = 12.3$$

$$\text{(ii)}\ \frac{c}{R} = \frac{1}{60} \quad (c = 8.3\times10^{-4}\ \text{m}), \quad \mu = 0.1\ \text{N s m}^{-2}; \quad \sigma_1 = 0.608, \quad \sigma_2 = 1.08.$$

Figures 6 and 7 show the neutral stability and operating curves for cases (i) and

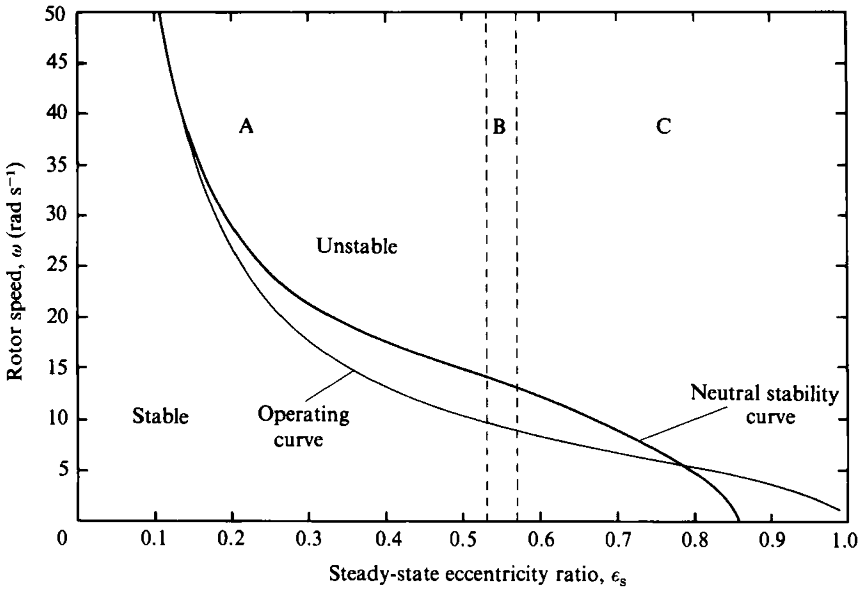

FIGURE 7. Neutral stability and operating curves ($\sigma_1 = 0.608$, $\sigma_2 = 1.08$).

(ii) respectively. In case (i) stable operation is predicted for shaft speeds in excess of approximately 2 rad s^{-1}. Furthermore the operating curve is well into the stable region at lower eccentricity ratios suggesting that such an experiment would not require great sensitivity. With case (ii) stable operation arises once the shaft speed reaches about 5 rad s^{-1} and remains so until approximately 40 rad s^{-1} (corresponding to an eccentricity ratio of just over 0.1) when the operating curve again approaches the neutral stability curve.

The assumption of no axial flow necessitates that in an experimental test the bearings would require end seals whose radial resistance to shaft movement, together with that presented by the coupling connecting the shaft to the drive motor, would have to be minimal. Such conditions have been met in previous experimental studies of bearing-influenced rotor dynamics. For the above cases the predicted minimum subambient pressures in the lubricant film are of the order -10^4 N m^{-2} gauge, for which cavitation would not be expected to occur. Such experiments would, therefore, present a clear opportunity to enable verification of the stabilizing action of fluid inertia in non-cavitating fluid-film bearings as predicted.

Appendix A

Necessary and sufficient conditions for all the roots of the real equation

$$A_0 \bar{n}^4 + A_1 \bar{n}^3 + A_2 \bar{n}^2 + A_3 \bar{n} + A_4 = 0$$

to have negative real parts are (Gantmacher 1959):

$$\text{(i) } A_0 > 0, \quad \text{(ii) } A_1 > 0, \quad \text{(iii) } A_3 > 0, \quad \text{(iv) } A_4 > 0,$$

$$\text{(v) } A_1 A_2 A_3 - A_1^2 A_4 - A_0 A_3^2 > 0.$$

On application to (4.20) these criteria reduce to the single requirement

$$a_0 \bar{\omega}^4 + a_1 \bar{\omega}^2 + a_2 > 0,$$

where the a_i (dependent upon ϵ_s only) are of order λ^i $(i = 0, 1, 2)$. For all admissible ϵ_s it can be shown that a_0 and a_1 are negative, and hence for $a_2 < 0$, $\bar{\omega}_c = 0$ whilst for $a_2 > 0$, $\bar{\omega}_c \sim \lambda^{\frac{1}{2}}$.

Appendix B

$$a_1 = S\left(\frac{\partial \bar{F}_X}{\partial X}\right)_s, \quad a_2 = S\left(\frac{\partial \bar{F}_X}{\partial Y}\right)_s, \quad a_3 = S\left(\frac{\partial \bar{F}_X}{\partial \dot{X}}\right)_s, \quad a_4 = S\left(\frac{\partial \bar{F}_X}{\partial \dot{Y}}\right)_s,$$

$$a_5 = S\left(\frac{\partial \bar{F}_X}{\partial \ddot{X}}\right)_s, \quad a_6 = S\left(\frac{\partial \bar{F}_X}{\partial \ddot{Y}}\right)_s, \quad a_{11} = S\left(\frac{\partial^2 \bar{F}_X}{\partial X^2}\right)_s, \quad a_{12} = S\left(\frac{\partial^2 \bar{F}_X}{\partial X \partial Y}\right)_s,$$

$$\ldots, \quad a_{111} = S\left(\frac{\partial^3 \bar{F}_X}{\partial X^3}\right)_s, \quad a_{112} = S\left(\frac{\partial^3 \bar{F}_X}{\partial X^2 \partial Y}\right)_s, \quad \ldots, \quad a_{666} = S\left(\frac{\partial^3 \bar{F}_X}{\partial \ddot{Y}^3}\right)_s.$$

These coefficients are evaluated to order λ, subject to the assumption that the double expansion of $\bar{F}_X$ and $\bar{F}_Y$ as a Taylor series and as a series in λ can be effected irrespective of order (see §4).

Order δ^3

$$\begin{aligned}
&\ddot{x}_3 - \bar{a}_1 x_3 - \bar{a}_2 y_3 - \bar{a}_4 \dot{x}_3 - \bar{a}_4 \dot{y}_3 - \bar{a}_5 \ddot{x}_3 - \bar{a}_6 \ddot{y}_3 \\
&\quad = -2\omega_1 \ddot{x}_2 - \omega_1^2 \ddot{x}_1 - 2\frac{\partial \dot{x}_1}{\partial \tau^*} + \bar{a}_3\left(\omega_1 \dot{x}_2 + \frac{\partial x_1}{\partial \tau^*}\right) \\
&\qquad + \bar{a}_4\left(\omega_1 \dot{y}_2 + \frac{\partial y_1}{\partial \tau^*}\right) + \bar{a}_5\left(2\omega_1 \ddot{x}_2 + \omega_1^2 \ddot{x}_1 + 2\frac{\partial \dot{x}_1}{\partial \tau^*}\right) \\
&\qquad + \bar{a}_6\left(2\omega_1 \ddot{y}_2 + \omega_1^2 \ddot{y}_1 + 2\frac{\partial \dot{y}_1}{\partial \tau^*}\right) + \mathrm{T}(\bar{a}_1)\, x_1 + \mathrm{T}(\bar{a}_2)\, y_1 + \mathrm{T}(\bar{a}_3)\, \dot{x}_1 \\
&\qquad + \mathrm{T}(\bar{a}_4)\, \dot{y}_1 + \mathrm{T}(\bar{a}_5)\, \ddot{x}_1 + \mathrm{T}(\bar{a}_6)\, \ddot{y}_1 + \bar{a}_{11} x_1 x_2 + \bar{a}_{12}(x_1 y_2 + x_2 y_1) \\
&\qquad + \bar{a}_{13}(x_1 \dot{x}_2 + x_2 \dot{x}_1 + \omega_1 x_1 \dot{x}_1) + \bar{a}_{14}(x_1 \dot{y}_2 + x_2 \dot{y}_1 + \omega_1 x_1 \dot{y}_1) \\
&\qquad + \bar{a}_{15}(x_1 \ddot{x}_2 + x_2 \ddot{x}_1 + 2\omega_1 x_1 \ddot{x}_1) + \bar{a}_{16}(x_1 \ddot{y}_2 + x_2 \ddot{y}_1 + 2\omega_1 x_1 \ddot{y}_1) \\
&\qquad + \bar{a}_{22} y_1 y_2 + \bar{a}_{23}(y_1 \dot{x}_2 + y_2 \dot{x}_1 + \omega_1 y_1 \dot{x}_1) + \bar{a}_{24}(y_1 \dot{y}_2 + y_2 \dot{y}_1 + \omega_1 y_1 \dot{y}_1) \\
&\qquad + a_{25}(y_1 \ddot{x}_2 + y_2 \ddot{x}_1 + 2\omega_1 y_1 \ddot{x}_1) + \bar{a}_{26}(y_1 \ddot{y}_2 + y_2 \ddot{y}_1 + 2\omega_1 y_1 \ddot{y}_1) \\
&\qquad + \bar{a}_{33}(\dot{x}_1 \dot{x}_2 + \omega_1 \dot{x}_1^2) + \bar{a}_{34}(\dot{x}_1 \dot{y}_2 + \dot{x}_2 \dot{y}_1 + 2\omega_1 \dot{x}_1 \dot{y}_1) + \bar{a}_{44}(\dot{y}_1 \dot{y}_2 + \omega_1 \dot{y}_1^2) \\
&\qquad + \tfrac{1}{6}\bar{a}_{111} x_1^3 + \tfrac{1}{2}\bar{a}_{112} x_1^2 y_1 + \tfrac{1}{2}\bar{a}_{113} x_1^2 \dot{x}_1 + \tfrac{1}{2}\bar{a}_{114} x_1^2 \dot{y}_1 + \tfrac{1}{2}\bar{a}_{115} x_1^2 \ddot{x}_1 + \tfrac{1}{2}\bar{a}_{116} x_1^2 \ddot{y}_1 \\
&\qquad + \tfrac{1}{2}\bar{a}_{122} x_1 y_1^2 + \bar{a}_{123} x_1 y_1 \dot{x}_1 + \bar{a}_{124} x_1 y_1 \dot{y}_1 + \bar{a}_{125} x_1 y_1 \ddot{x}_1 + \bar{a}_{126} x_1 y_1 \ddot{y}_1 \\
&\qquad + \tfrac{1}{2}\bar{a}_{133} x_1 \dot{x}_1^2 + \bar{a}_{134} x_1 \dot{x}_1 \dot{y}_1 + \tfrac{1}{2}\bar{a}_{144} x_1 \dot{y}_1^2 + \tfrac{1}{6}\bar{a}_{222} y_1^3 + \tfrac{1}{2}\bar{a}_{223} y_1^2 \dot{x}_1 \\
&\qquad + \tfrac{1}{2}\bar{a}_{224} y_1^2 \dot{y}_1 + \tfrac{1}{2}\bar{a}_{225} y_1^2 \ddot{x}_1 + \tfrac{1}{2}\bar{a}_{226} y_1^2 \ddot{y}_1 + \tfrac{1}{2}\bar{a}_{233} y_1 \dot{x}_1^2 + \bar{a}_{234} y_1 \dot{x}_1 \dot{y}_1 \\
&\qquad + \tfrac{1}{2}\bar{a}_{244} y_1 \dot{y}_1^2,
\end{aligned} \tag{B 1}$$

where the operator $\mathrm{T}(\bar{a})$ is defined by:

$$\mathrm{T}(\bar{a}) = -\frac{2\bar{a}}{\bar{\omega}_c} + \frac{\mathrm{d}\bar{a}}{\mathrm{d}\bar{\omega}}.$$

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